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SPECIAL TOPICS



The plasma resonance in the response and in the rf impedance of a capacitively shunted Josephson junction in the presence of thermal noise

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The rf impedance and the wide-band response to radiation of a resistively shunted junction (RSJ) model Josephson junction have been measured in the presence of thermal noise, using a phase-locked-loop analog of the RSJ model. In the conditions for which analytical calculations are valid there is good agreement between the theory and the analog. When the RSJ model is shunted with a capacitance, a plasma type resonance can occur in the rf impedance when the junction is biased in the supercurrent (in-lock). When thermal noise is present, this plasma resonance can also occur when a nonzero average voltage is generated across the junction. We found that in this case a resonance also occurs in the wide-band response to radiation. This resonance takes place at a frequency that can be identified as the attempt frequency, with which the junction attempts to escape the phase locked condition in which it exists for a fraction of the time even though the average voltage across the junction is nonzero. The average lifetimes of the junction in lock τ_{in} ($\bar{v} = 0$) and out of lock τ_{out} ($\bar{v} = \bar{v}_{out}$) were also measured in the presence of thermal noise. τ_{in} is in good agreement with existing theory and τ_{out} is derived from the measurements.

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I. INTRODUCTION

The Josephson differential equation¹ can be solved analytically only for a very limited number of situations. When interested in the general behavior of a Josephson junction, either a numerical approach or the use of an analog is required.

As was shown by Bak *et al.*^{2,3} a close analogy exists between the resistively shunted junction (RSJ) model of a Josephson junction and a phase-locked loop (PLL).

We used such an analog to find the solutions of the Josephson differential equation, when rf signal currents and thermal noise are present and when a shunting capacitance is added to the RSJ model. For a capacitance C such that McCumber's damping parameter $\beta_c = 2eI_c R^2 C / \hbar \gtrsim 1$ (where I_c is the maximum supercurrent and R is the normal state resistance)⁴ there exists a plasma resonance in the rf impedance of a Josephson junction, when it is biased in the supercurrent with the average voltage \bar{V} across the junction equal to zero.^{1,5,6} Our main interest was to investigate the possible occurrence of plasma effects in the rf impedance and the wide-band response to radiation, in the presence of thermal noise and for $\bar{V} \neq 0$, as was proposed by Tolner.⁷

We therefore used a PLL analog to measure the rf impedance and the wide-band response to radiation in the case of a dc and rf current driven junction, both with and without a capacitive shunt and in the presence of thermal noise. The time-dependent behavior of the voltage across a junction with $\beta_c \gtrsim 1$ in the presence of thermal noise was also investigated.

In Sec. II a description is given of the analog and of its quality as a simulation for the RSJ model. In Sec. III the

measured rf impedance and the wide-band response are given of a junction without a capacitive shunt ($\beta_c = 0$), but in the presence of thermal noise. A comparison is made between the measurements and existing theory. In Sec. IV the behavior of the RSJ model is described when it is shunted by a capacitance ($\beta_c > 1$) in the presence of thermal noise. The resulting impedance and response curves in this case are presented in Sec. V.

II. THE ANALOG

A. Introduction

As is well known,¹ the current through an ideal resistively shunted Josephson junction (RSJ model) is given by

$$I = I_c \sin \phi + V/R, \quad (1)$$

with

$$V = (\hbar/2e) d\phi / dt. \quad (2)$$

ϕ is the phase difference in the order parameters on each side of the barrier; I_c is the maximum pair current (of Cooper pairs); V is the voltage generated across the junction; and R is the normal state resistance of the junction, which is supposed to be independent of V . Combining Eqs. (1) and (2) results in

$$I = I_c \sin \phi + (\hbar/2eR) d\phi / dt. \quad (3)$$

The quasi-particle-pair interference current¹ ($\cos \phi$ term) can be included by multiplying the quasi-particle current V/R by a factor $(1 + \alpha \cos \phi)$, where α is a function of temperature and voltage. This results in

$$I = I_c \sin \phi + (\hbar/2eR) d\phi / dt + \alpha \cos \phi (\hbar/2eR) d\phi / dt. \quad (4)$$

As was first described by Bak *et al.*^{2,3} a phase-locked loop (PLL) is described by an equation of the same form as Eq. (3), thereby representing an analog for the ideal RSJ model.

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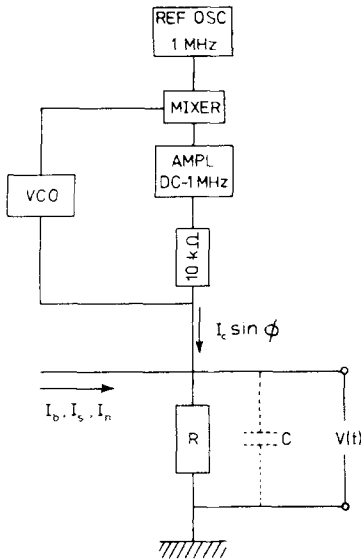


FIG. 1. Block diagram of the phase-locked-loop analog. As voltage controlled oscillator (VCO) a HP3301A is used. $I_c = 170 \mu\text{A}$, $R = 100 \Omega$, $k_{\text{vco}} = 2e/h = 447 \text{ kHz/V}$. The shunting capacitance shown by the dashed lines will be introduced later.

In our case a PLL is obtained (Fig. 1) by using a voltage controlled oscillator (VCO) and a reference oscillator, both at 1 MHz. The difference signal obtained by mixing the signals from the two oscillators is used to control the VCO. A current, proportional to the difference signal, is injected into the resistor $R = 100 \Omega$ at the input of the VCO. A quasicurrent injection is obtained, using a 10-k Ω series resistance. In this way, the VCO is locked to the reference oscillator. When a dc current I_b is injected into R , the PLL will react by generating a constant phase difference ϕ between the VCO and the reference oscillator, resulting in a dc current $I_c \sin \phi$ through R that cancels I_b . Therefore no voltage is developed across R when phase lock is maintained, as for a Josephson junction. I_c depends on the loop gain, the oscillator power levels, and the VCO constant k_{vco} . The maximum I_b for which the PLL maintains locking is equal to $I_b = \pm I_c$, when $\phi = \pm \pi/2$. When $|I_b| > I_c$, the phase difference ϕ is no longer constant, and nonlinear oscillations occur at a radial frequency $d\phi/dt = 2\pi k_{\text{vco}} V$, resulting in the same current-voltage behavior as a Josephson junction. The VCO constant k_{vco} replaces $2e/h$ in the Josephson equations [see Eq. (2)]. The deviation of the simulated pair current, from the ideal $I_c \sin \phi$ behavior is in our case less than 4%.

Where it is useful we will, from now on, use normalized parameters to simplify the equations. They will be denoted by lower case characters, except for Ω and Γ which are the normalized parameters for frequency and noise level. Currents are divided by I_c , voltages by $I_c R$, and frequencies by $\omega_0 = 2eI_c R / \hbar$.

According to Bak *et al.*^{2,3} the quasi-particle-pair interference current in a Josephson junction can be simulated in the analog by limiting the bandwidth of the PLL. The closed-loop input admittance Y of the PLL analog without a capacitive shunt and in case of phase locking ($\dot{\phi} = 0$) is then²

$$Y = \frac{\cos \phi}{j\Omega R [1 + (\Omega / \Omega_c)^2]} + \frac{1}{R} - \frac{\cos \phi}{R\Omega_c [1 + (\Omega / \Omega_c)^2]}, \quad (5)$$

where Ω_c is the normalized cutoff frequency of the loop $\Omega_c = \omega_c / \omega_0$, with $\omega_0 = 2eI_c R / \hbar$ and $\Omega = \omega / \omega_0$. The admittance of the RSJ model including the $\cos \phi$ term, when biased in the supercurrent ($\bar{v} = 0$), can easily be calculated from Eqs. (2) and (4) and is given by

$$Y = \frac{\cos \phi}{j\Omega R} + \frac{1}{R} + \frac{\alpha \cos \phi}{R}. \quad (6)$$

The close relation between the admittance of the analog and the RSJ model is clear. The last terms of Eqs. (5) and (6) represent the quasi-particle-pair interference term with $\alpha = -\{\Omega_c [1 + (\Omega / \Omega_c)^2]\}^{-1}$. Notice that α is frequency dependent in the analog. The first term in Eq. (6) represents the pair conductivity, while the corresponding term in the analog differs from this by a factor $[1 + (\Omega / \Omega_c)^2]^{-1}$. This means that the pair conductivity in the analog is considerably different from that in the RSJ model for $\Omega \gtrsim 0.2\Omega_c$. Therefore the PLL analog can only be used to simulate the RSJ model including the $\cos \phi$ term in a very limited frequency interval with $\Omega \ll \Omega_c$. In that case $\alpha = -(\Omega_c)^{-1}$ and $\alpha = -1$ can easily be reached. However, we are interested in the behavior of the RSJ model over a relatively large frequency interval $0 < \Omega < 1.5$, which is only possible by choosing $\Omega_c \gg 1$, so that $\alpha \simeq 0$. In our model the minimum value for α is limited by the cutoff frequency of the VCO of about $\Omega_c = 10$ so that $|\alpha| \lesssim 0.1$. This value of α is small enough to be neglected. Our model therefore does not include the quasi-particle-pair interference current.

B. Noise

In a Josephson junction we can define⁸ a dimensionless noise parameter

$$\Gamma = 2ek_B T_e / \hbar I_c, \quad (7)$$

where T_e is the effective noise temperature which is, in the ideal case, equal to the physical temperature of the junction. The noise can be thought to originate from the resistance R at a temperature T_e . This noise can be represented⁹ by a noise current generator parallel with R and, in this case, also parallel with the pair conductivity and possibly a shunting capacitance. This current has an amplitude

$$I_{N_{\text{RMS}}} = (4k_B T_e B / R)^{1/2}, \quad (8)$$

where B is the effective noise bandwidth. From Eq. (8) we can obtain an effective noise temperature T_e , where

$$T_e = I_{N_{\text{RMS}}}^2 R / 4k_B B. \quad (9)$$

Combining Eqs. (7) and (9) we obtain

$$\Gamma = \frac{eR}{2\hbar I_c B} I_{N_{\text{RMS}}}^2 = \frac{eI_c R}{2\hbar B} i_{n_{\text{RMS}}}^2 = \frac{\pi}{2b} i_{n_{\text{RMS}}}^2, \quad (10)$$

where $i_{n_{\text{RMS}}} = I_{N_{\text{RMS}}} / I_c$ is the normalized RMS noise current through the junction and b is the normalized effective noise bandwidth $b = 2\pi B / \omega_0$.

In the analog this noise current $I_{N_{\text{RMS}}}$ can be easily simulated by external injection. To simulate thermal noise, it is

necessary that the frequency spectrum of the injected noise is flat up to frequencies much higher than $\omega_0/2\pi = 2eI_c R/h$, giving the order of magnitude of the highest frequencies generated in the junction. In our analog $b = 6.3 \pm 0.3$, which is indeed much larger than one ($0 \leq \Omega_n \leq 6.3$).

As can be seen from Eq. (10), Γ is quadratically dependent on $I_{N_{RMS}}$, in contrast with the so-called effective peak-to-peak noise current I_{ph} , as defined by Likharev *et al.*¹⁰ $I_{ph} \equiv 2I_c \Gamma$. Taking I_{ph} as a measure for the relative noise currents through the junction leads to a dramatic underestimate of the effects of noise on the I - V curve and on the ratio of the signal currents to the noise currents for $\Gamma \ll 1$. Since only noise currents with normalized frequencies lower than about one have an effect on the junction,¹⁰ $b_{eff} \simeq 1$, so that from Eq. (10) $\Gamma \simeq (1.25 I_{N_{RMS}}/I_c)^2$ is obtained. Then for $\Gamma = 1$, $I_{ph} = 2.5 I_{N_{RMS}} = 2I_c$; while for $\Gamma = 0.01$, $I_{ph} = 0.25 I_{N_{RMS}} = 0.02 I_c$. The peak-to-peak noise current is about $6 I_{N_{RMS}}$ so that the difference between I_{ph} and the actual peak-to-peak noise currents in the junction can be very large for small Γ values. At a certain noise level, the junction can therefore be thrown "out of lock" by noise at considerably lower values of the bias current than is expected from the size of I_{ph} .

C. Capacitive shunting

The influence of a capacitive shunt on the behavior of a Josephson junction is represented by adding an extra term to Eq. (3). The behavior of the phase difference ϕ is then given by

$$I = I_c \sin \phi + (\hbar/2eR) d\phi/dt + C(\hbar/2e) d^2\phi/dt^2. \quad (11)$$

In the analog this can be simulated directly by shunting the resistance R with a capacitance C , as shown in Fig. 1.

D. Comparison between the analog and the RSJ model

To obtain an idea of the quality of the PLL analog as a simulation of the RSJ model, we measured a number of parameters that have been calculated in the literature.

(1) For $\beta_c = 0$ (without a capacitive shunt) the maximum normalized differential resistance $(d\bar{v}/d\bar{i})_{max}$ and the normalized average voltage \bar{v}_0 at which this maximum occurs were measured as a function of Γ ($\bar{v} = \bar{V}/I_c R$ and $\bar{i} = \bar{I}_s/I_c$). As is shown later in Fig. 9, the measured values for $[(d\bar{v}/d\bar{i})_{max} - 1]$ are only slightly higher than given by Likharev *et al.*,¹¹ while a good fit is obtained for the corresponding values of \bar{v}_0 .

(2) The size of the hysteresis in the I - V curve (for $\Gamma = 0$) as a function of β_c is in agreement with the hysteresis first calculated by Stewart¹² and McCumber⁴ (not shown). The effect of the $\cos \phi$ term on the hysteresis has been given by Auracher *et al.*¹³ A comparison (not shown) reveals that in our analog this effect is negligibly small, confirming that α is indeed much smaller than one.

(3) The I - V curves obtained with the analog are in good agreement with the curves computed by Ambegaokar *et al.*¹⁴ for $\beta_c = 0$ and by Kurkijärvi *et al.*¹⁵ for $\beta_c = 1$ and 4 for $\Gamma = 0.05$ (not shown).

(4) The dependence of the step heights induced in the I - V curves by rf monochromatic currents, on the amplitude and frequency of the currents, is in good agreement with calculations by Likharev *et al.*¹⁶ for $\Omega < 1$ and with a Bessel function behavior for $\Omega > 1$ (not shown).

(5) The rf impedance for $\beta_c = 0$ without thermal noise, is in good agreement with the calculations by Auracher *et al.*¹⁷ (not shown), both in the supercurrent ($\bar{v} = 0$) and for $\bar{i} > 1$ ($\bar{v} \neq 0$). The main difference is the occurrence of small subharmonic singularities at reduced voltages $\bar{v} = 1/n$, where $n = 2, 3, 4$, etc.

In conclusion we can say that these measurements give reasonable confidence in the applicability of the analog to cases where a capacitive shunt and thermal noise are present at the same time.

III. PLL ANALOG WITHOUT A CAPACITIVE SHUNT

$\beta_c = 0$

A. rf Impedance in the presence of thermal noise

The rf impedance of a Josephson junction in the presence of low-level thermal noise has been calculated by Vystavkin *et al.*⁸ A comparison will be made between the impedance as measured with the analog and these calculations. The impedance measurements will also be extended beyond the constraints on the validity of the calculations.

Figure 2 shows the measured normalized junction impedance $\text{Re}(z) = \text{Re}(Z)/R$ and $\text{Im}(z) = \text{Im}(Z)/R$ (where Z is the junction impedance) versus Ω for different values of \bar{v} at $\Gamma = 0.04$. This Γ value is chosen as a representative value for high Ohmic junctions e.g., an effective noise temperature $T_e = 5$ K, $R = 200 \Omega$, and typically $I_c = 5 \mu\text{A}$ corresponds, to $\Gamma \simeq 0.04$. The rf signal current $i_s = I_s/I_c = 0.1$ is small enough to avoid saturation.

The impedance around the singularity (at $\bar{v} = \Omega$) and in the presence of thermal noise, has been calculated by Vystavkin *et al.*⁸ and is given in reduced units by

$$z = 1 - \frac{1}{2(\bar{i} + \bar{v})} \frac{\Delta + j\Gamma_1}{\Delta^2 + \Gamma_1^2}, \quad |\Delta|, \Gamma_1 \ll \Omega, \quad (12)$$

where $\Delta = \Omega - \bar{v}$ and $\Gamma_1 = (1 + 1/2\bar{i}^2)(d\bar{v}/d\bar{i})^2\Gamma$. This impedance is shown in Fig. 2 by dashed-dotted lines for $\bar{v} = 0.8$. The maximum in the measured $\text{Re}(z)$ (at $\Delta \simeq -\Gamma_1$) is in good agreement with Eq. (12), while the minimum (at $\Delta \simeq \Gamma_1$) is somewhat below the theoretical value. For $|\Delta| \gtrsim 0.4$, where Eq. (12) is not strictly valid, the difference increases. The $\text{Im}(z)$ as given by Eq. (12) is in good agreement with the measurements for $\Omega \leq 0.8$ [also in the region where Eq. (12) is not valid]. For $\Omega > 0.8$ the measured $\text{Im}(z)$ is higher than given by Eq. (12). From the figure it is clear that this difference is caused by the fact that the impedance for $\bar{v} \neq 0$ is also determined by the impedance for $\bar{v} = 0$. The junction is "in lock" ($\bar{v} = 0$) part of the time even though $\bar{v} \neq 0$. This effect becomes stronger for smaller \bar{v} values, where the reactance is inductive for a large frequency interval. Apparently only the high-frequency side of the smoothed singularity is affected by the $\bar{v} = 0$ behavior. The measured impedance for $\bar{v} = 0$ is in good agreement with the in-lock impedance which is given

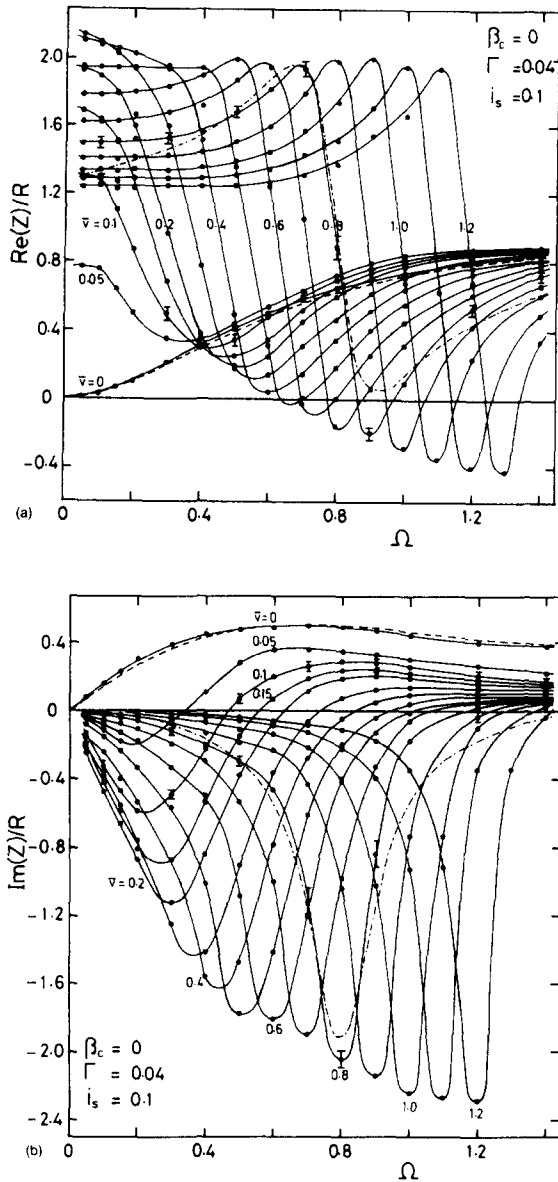


FIG. 2. Measured normalized real (a) and imaginary (b) part of the impedance of an RSJ model Josephson junction, as a function of normalized frequency and for several values of \bar{v} . The dashed lines show the impedance as calculated from Eq. (6) with $\alpha = 0$ for $\bar{v} = 0$ and without noise, but for the same value of \bar{i} . The dashed-dotted lines are calculated using Eq. (12), for $\bar{v} = 0.8$.

by R shunted by the Josephson inductance $L_J = (\hbar/2eI_c \cos\phi)^{-1}$ (see dashed lines in Fig. 2).

B. Response to radiation in the presence of thermal noise

As was first shown by Kanter *et al.*¹⁸ the wide-band response to an rf current $\Delta\bar{v}/i_s^2$ of an RSJ junction without noise is given (in normalized units) by

$$\Delta\bar{v}/i_s^2 = (d\bar{v}/d\bar{i})/4\bar{i}\Omega^2, \quad \text{for } \bar{v} \ll \Omega, \quad (13a)$$

$$\Delta\bar{v}/i_s^2 = (d^2\bar{v}/d\bar{i}^2)/4, \quad \text{for } \bar{v} \gg \Omega, \quad (13b)$$

$$\Delta\bar{v}/i_s^2 = (d\bar{v}/d\bar{i})/8\bar{i}\Omega\Delta, \quad \text{for } \bar{v} \simeq \Omega. \quad (13c)$$

In the presence of thermal noise, Eqs. (13a) and (13b) can still be used when the junction is biased at $\bar{v} > \bar{v}_0$, as was shown by

Likharev *et al.*¹⁰ For smaller bias voltages, deviations occur and a solution is only possible using the Einstein-Fokker-Planck (EFP) equation.¹⁰ With noise the singularity in Eq. (13c), for $\Delta = 0$, disappears. For sufficiently small detuning ($\Delta \ll \Omega$) and for a low level of noise, ($\Gamma_1 \ll \Omega$), the response in this case can be given by Eq. (14), which is the real part of Eq. (13c) with Δ replaced by $\Delta + j\Gamma_1$.^{8,10}

$$\Delta\bar{v}/i_s^2 = \frac{(d\bar{v}/d\bar{i})}{8\bar{i}\Omega} \frac{\Delta}{\Delta^2 + \Gamma_1^2}, \quad |\Delta|, \Gamma_1 \ll \Omega. \quad (14)$$

For $\bar{v} > \bar{v}_0$, the differential resistance is $d\bar{v}/d\bar{i} = \bar{i}/\bar{v}$ and since $|\Delta| \ll \Omega$, the average voltage $\bar{v} \simeq \Omega$ so that

$$\bar{v}/i_s^2 = \Delta/[8\Omega^2(\Delta^2 + \Gamma_1^2)], \quad |\Delta|, \Gamma_1 \ll \Omega, \quad (15)$$

as given already by Vystavkin *et al.*⁸ According to the same reference Eqs. (13a) and (13b) are also valid for the less restricted cases of $\bar{v} < \Omega$ and $\bar{v} > \Omega$, respectively, when thermal noise is included.

Using the PLL analog we measured the response to an rf current $\Delta\bar{v}/i_s^2$ of a dc current biased junction in the small signal limit ($i_s = 0.1$) and in the presence of simulated thermal noise. In Fig. 3 the measured response for $\bar{v} = 0.8$ and $\bar{v} = 0.3$ is compared with Eqs. (13a), (13b), and 14 for $\Gamma = 0.04$. For $\bar{v} = 0.8$ [Fig. 3(a)] the response for $\Omega \gg \bar{v}$, $\Omega \simeq \bar{v}$, and $\Omega \ll \bar{v}$ is in good agreement with Eqs. (13a), (14), and (13b), respectively. For $\frac{1}{2}\bar{v} \lesssim \Omega \lesssim \bar{v}$ the response consists of both the classical response and the (smoothed) singular re-

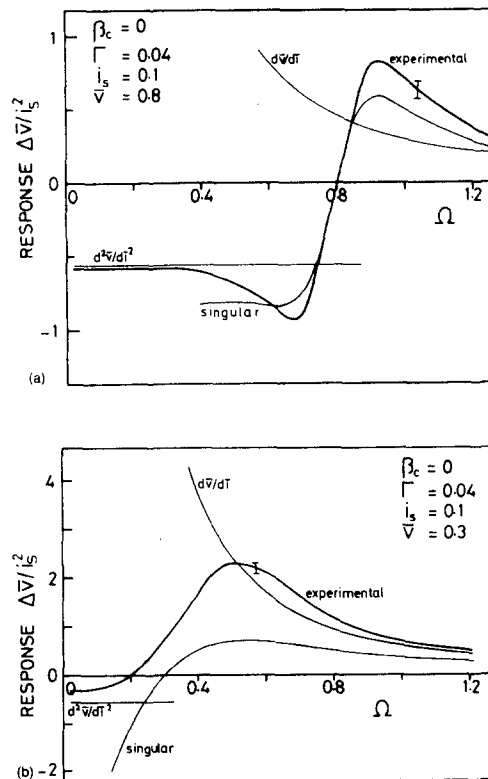


FIG. 3. Comparison for $\beta_c = 0$, between the measured normalized response (as a function of normalized frequency), at $\bar{v} = 0.8$ (a) and $\bar{v} = 0.3$ (b), and the response calculated using Eqs. (13a), (13b), and (14) with the corresponding experimental values of $d\bar{v}/d\bar{i}$ and $d^2\bar{v}/d\bar{i}^2$.

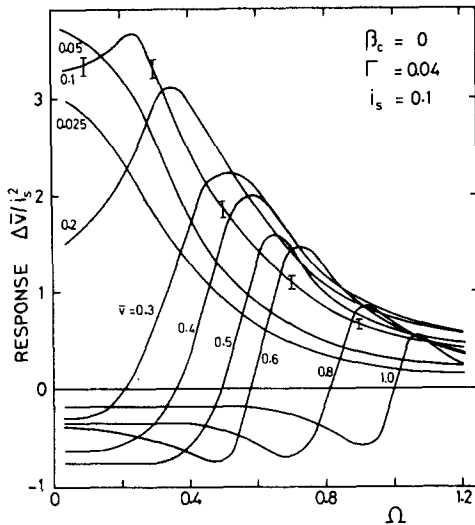


FIG. 4. Measured normalized response to radiation as a function of normalized frequency for several values of $\bar{\nu}$, for $\beta_c = 0$.

sponse, while for $\bar{\nu} \lesssim \Omega \lesssim 2\bar{\nu}$ the response consists of both the response of Eq. (13a) and the (smoothed) singular response. The measurements on the analog show how these different response types add in these frequency intervals. For $\bar{\nu} = 0.3$ [Fig. 3(b)] only for $\Omega \gg \bar{\nu}$ and $\Omega \ll \bar{\nu}$ the response is in good agreement with Eqs. (13a) and (13b), while in the region in between, Eq. (14) is invalid because $\Gamma_1 = 0.4 > \Omega$.

From this comparison we can conclude that the measurements with the analog are in good agreement with the analytical expressions within their regions of validity.

Figure 4 shows the measured response as a function of Ω for different values of $\bar{\nu}$ and at $\Gamma = 0.04$. The response is at a maximum for $0.05 \lesssim \bar{\nu} \lesssim 0.1$ and $0.05 \lesssim \Omega \lesssim 0.3$, where the analytical expressions are invalid so that the response can only be calculated numerically, using the EFP equation.

The current response shown in Fig. 4 is similar to the response calculated by Zavaleyev *et al.*,¹⁹ where a Josephson junction is coupled to an external system. In that case, however, Zavaleyev finds a maximum response for $\Gamma \simeq 0.4$, while

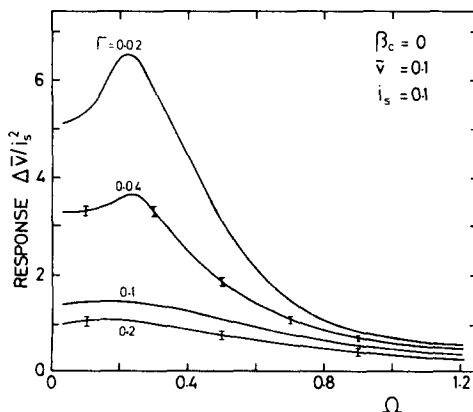


FIG. 5. Measured normalized response ($\beta_c = 0$) as a function of Ω , for $\bar{\nu} = 0.1$, and for several values of Γ .

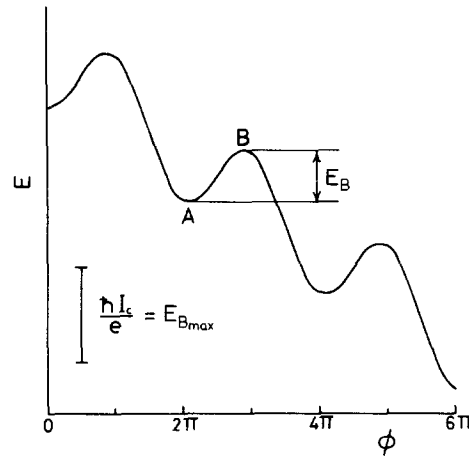


FIG. 6. Potential energy E vs the phase difference across the barrier, for a current biased Josephson junction. The potential energy difference E_B between A and B is the energy barrier.

we find a response that increases with decreasing Γ at least down to $\Gamma = 0.005$ (not shown).

For later comparison with the response for nonzero β_c values we present in Fig. 5 the response versus Ω for $\bar{\nu} = 0.1$ for several Γ values. $\Gamma = 0.02$ is the smallest value possible for Γ , to avoid saturation at a signal current of $i_s = 0.1$. For $\Gamma > 0.1$, maximum response occurs for $\bar{\nu} \simeq 0.1$ at all frequencies.

IV. CAPACITIVE SHUNTING IN THE PRESENCE OF THERMAL NOISE $\beta_c \neq 0$

A. Dynamics

In order to understand the effect of a capacitive shunt in the presence of thermal noise we will first look into the dynamics of a Josephson junction.

When a shunting capacitance is added to the RSJ model, the I - V curve without thermal noise becomes hysteretic for $\beta_c \gtrsim 0.8$.^{4,12} Within the hysteretic region the differential equation (11), describing the time-dependent behavior of ϕ in this case, has two solutions. These solutions can be visualized in the "washing board" analog⁵ shown in Fig. 6: A particle moving in a potential energy $E = (-i\phi - \cos\phi)$, where ϕ is the particle position and corresponds with the phase difference across the junction, and i is the slope of the washing board and corresponds with the instantaneous current through the junction.

When $\bar{i} = 0$, the particle is at the bottom of the potential well with $\phi = 0$, the junction is in lock. Upon increasing \bar{i} , the average phase increases until, beyond $\bar{i} = 1$, the particle rolls down the washing board continuously. When, for $\beta_c \gtrsim 0.8$, the bias current is within the hysteretic region $\bar{i}_{\min} < \bar{i} < 1$ (\bar{i}_{\min} is the lower boundary of the interval) and \bar{i} is momentarily increased above one, the particle rolls down the next potential well. Then the increase of the kinetic energy of the particle in one period of ϕ is larger than the binding energy E_B , so that the particle is not captured in this next potential well. Only for $\bar{i} < \bar{i}_{\min}$ the particle is recaptured.

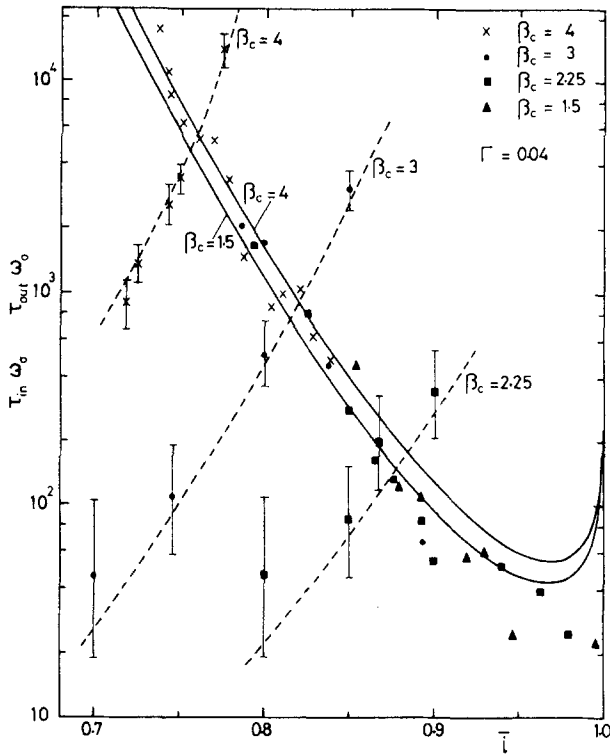


FIG. 7. Measured normalized average lifetimes, in lock $\tau_{in}\omega_0$ (without error bars) and out of lock $\tau_{out}\omega_0$ (with error bars), as a function of the bias current, for several values of β_c . The solid lines are obtained using Eq. (19) for $\tau_{in}\omega_0$ and the dashed lines result from using Eq. (21) for $\tau_{out}\omega_0$.

In the presence of thermal noise, however, the phase ϕ can be switched out of lock even for $\tilde{i} < 1$, while it can also be captured for $\tilde{i} > \tilde{i}_{min}$. As has been shown by Fulton *et al.*,²⁰ the average lifetime of the phase-lock situation τ_{in} is given by

$$\tau_{in} = \nu_a^{-1} \exp(E_B/k_B T_e), \quad (16)$$

where ν_a is the attempt frequency with which the particle attempts to escape from the potential well, E_B is the energy barrier indicated in Fig. 6, and T_e is the effective temperature of the junction. The attempt frequency has already been calculated by Kramers.²¹ For a Josephson junction it is given in reduced units by⁷

$$\nu_a(2\pi/\omega_0) = \Omega_a = (2\beta_c)^{-1} [(1 + 4\beta_c \cos\phi)^{1/2} - 1]. \quad (17)$$

For large β_c values Ω_a reduces to the plasma frequency $\Omega_p = (\cos\phi/\beta_c)^{1/2}$ (see Sec. V). The energy barrier E_B is given by²⁰

$$E_B = (\hbar I_c/2e) [\tilde{i} 2 \sin^{-1}(\tilde{i} - \pi) + 2 \cos(\sin^{-1}\tilde{i})] = E_{B_{max}} f(\tilde{i}), \quad (18)$$

where $E_{B_{max}} = \hbar I_c/e$ is the maximum binding energy.

With the PLL analog we measured the average lifetime of the junction, when in lock τ_{in} , as a function of \tilde{i} , for $\beta_c = 1.5, 2.25, 3$, and 4 and $\Gamma = 0.04$. The result is shown in Fig. 7 (data points without error bars), where the vertical scale is in units ω_0^{-1} . The solid lines for $\beta_c = 1.5$ and 4 are obtained using Eq. (16), which can also be written as

$$\tau_{in}\omega_0 = 2\pi\Omega_a^{-1} \exp[2f(\tilde{i})/\Gamma]. \quad (19)$$

This equation also shows the strong dependence of τ_{in} on Γ . In fact the measurements of τ_{in} yield an accurate calibration of Γ . For $\Gamma = 0.040 \pm 0.002$ we obtain a good agreement between Eq. (19) and the measured values of τ_{in} for the four different values of β_c .

We now consider the situation out of lock, where no expression for the average lifetime is available. Out of lock (when in the mechanical analog the particle is rolling down the washing board [see Fig. 6]) the kinetic energy represents a barrier against recapture of the particle. It is clear that a local minimum in the kinetic energy of the particle occurs when it is at a local maximum of the potential energy. If this kinetic energy is zero, then the particle will be recaptured. In the noiseless situation this occurs for $\tilde{i} = \tilde{i}_{min}$ (see above). With thermal noise the particle will also be recaptured for $\tilde{i} > \tilde{i}_{min}$.

With the PLL analog we measured the average lifetimes out of lock τ_{out} , which have also been plotted in Fig. 7 (data points with error bars), for $\beta_c = 2.25, 3$, and 4 . Similar to excitation from the phase-locked condition, the results for τ_{out} can be understood by thermal excitation from the rotating phase condition. The barrier height in this case is the minimum electrostatic energy of the junction (similar to the minimum kinetic energy of the particle in the mechanical analog) during one revolution of the phase (a phase slip of 2π),

$$E_c = CV_{min}^2(\tilde{i}, \beta_c)/2 = \beta_c E_{B_{max}} v_{min}^2(\tilde{i}, \beta_c)/4, \quad (20)$$

where $V_{min}(\tilde{i}, \beta_c)$ is the minimum voltage across the junction during one revolution of ϕ in the noiseless situation; V_{min} depends strongly on β_c . And *ad hoc* attempt frequency can be taken equal to the Josephson oscillation frequency $\nu = 2e\bar{V}_{out}(\tilde{i}, \beta_c)/h = (\omega_0/2\pi)\bar{v}_{out}(\tilde{i}, \beta_c)$, where $\bar{V}_{out}(\tilde{i}, \beta_c)$ is the out-of-lock average voltage. Then similar to Eq. (19)

$$\tau_{out}\omega_0 = 2\pi\bar{v}_{out}^{-1}(\tilde{i}, \beta_c) \exp[\beta_c v_{min}^2(\tilde{i}, \beta_c)/2g\Gamma]. \quad (21)$$

If β_c is large, $\bar{v}_{out}(\tilde{i}, \beta_c) \simeq i$, independent from β_c . g is an unknown factor taking into account the effective noise level; because of mixing of high-frequency noise components, $g > 1$. By measuring $v_{min}(\tilde{i}, \beta_c)$ and $\bar{v}_{out}(\tilde{i}, \beta_c)$ for different β_c values we find that a match exists between the measured values of τ_{out} and Eq. (21) when $g = 1.39 \pm 0.03$. Equation (21) is shown in Fig. 7 by dashed lines for three β_c values. For $\beta_c = 1.5$ the uncertainties are too large to show a significant correlation between the measurements and Eq. (21) (not shown).

B. Differential resistance

At a sufficiently high thermal noise level, the I - V curves for $\beta_c \gtrsim 0.8$ become nonhysteretic, i.e., for each setting of the bias current there is a single-valued time-averaged voltage. For $\Gamma > 0.2$ the I - V curve is nonhysteretic for almost any β_c value. As an example Fig. 8 shows the I - V curves for the case of $\beta_c = 100$, for three different Γ values, compared with the hysteretic I - V curve without noise. Contrary to the case of $\beta_c = 0$, however, the voltage across the junction no longer consists of a continuous series of pulses, but of pulse trains (with average duration τ_{out}), separated by an average "dead

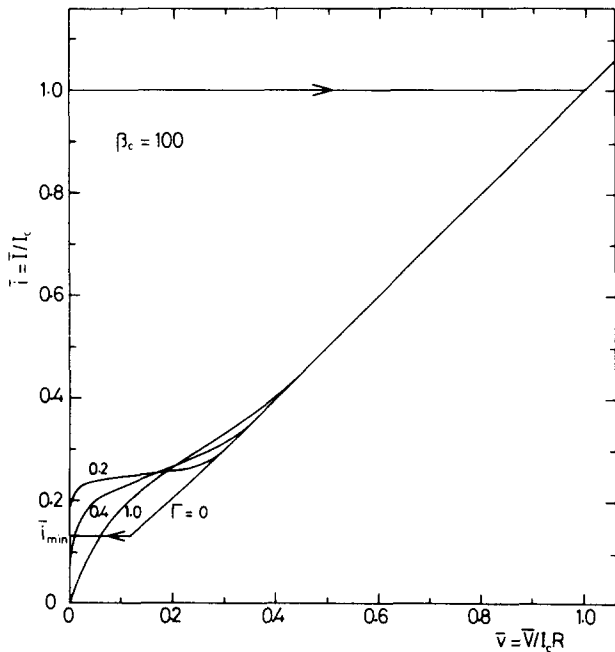


FIG. 8. Measured I - V curves for $\beta_c = 100$. A comparison between the noiseless situation (hysteretic, $\Gamma = 0$) and the case with thermal noise present (nonhysteretic, $\Gamma = 0.2, 0.4$, and 1.0).

time" τ_{in} . A quasistationary I - V curve therefore only occurs when the voltage is averaged over a time τ , where $\tau \gg \tau_{in}$ and $\tau \gg \tau_{out}$.

The differential resistance $d\bar{v}/d\bar{i}$ was measured (with the PLL analog) as a function of \bar{v} for different β_c and Γ values. Two different methods were used, depending on the values of β_c and Γ . For $\beta_c > 1$ in the case $(d\bar{v}/d\bar{i})_{max} > 10$ and for $\beta_c \geq 20$, the differential resistance is obtained by estimating the slope of the I - V curve (e.g., see Fig. 8). For all other combinations of β_c and Γ , the bias current is modulated with a normalized frequency $\Omega_{mod} = 5 \times 10^{-3}$ and the resulting voltage across the junction is measured. A calibration is obtained in both cases by a measurement for $\bar{i} \gg 1$. For large β_c values and small Γ , it is not possible to measure a voltage across the junction, with a normalized frequency of 5×10^{-3} , because the average lifetimes τ_{in} and τ_{out} become longer than one period of the modulating frequency.

The maximum values $[(d\bar{v}/d\bar{i})_{max} - 1]$ and the voltages \bar{v}_0 at which they occur, have been plotted in Fig. 9 as a function of Γ . The dashed lines were calculated by Likharev *et al.*¹¹ for $\beta_c = 0$. From Fig. 9 it is clear that $(d\bar{v}/d\bar{i})_{max}$ increases with increasing β_c , but that saturation occurs for high β_c values.

V. THE PLASMA RESONANCE

From the previous discussion it is clear that an underdamped RSJ junction, with an I - V curve smoothed by thermal noise, can be biased in such a way that the junction is in lock for a large fraction of the time even when $\bar{v} \neq 0$.

When biased in the supercurrent, the pair conductivity can be represented by an inductance $L_J = \hbar(2eI_c \cos \phi)^{-1}$ (parallel with the quasi-particle resistance R) that can form a resonant system with a shunting capacitance C at

$$\Omega_p = (\cos \phi / \beta_c)^{1/2}. \quad (22)$$

Microscopically this is the well-known plasma resonance in the supercurrent^{1,5} which was first observed experimentally by Dahm *et al.*⁶ Now if the junction is in-lock part of the time, even when $\bar{v} \neq 0$, plasma effects should also occur in this case, and should be observable in the junction impedance and in the response to external radiation. According to Tolner⁷ thermal excitation at the attempt frequency (Sec. IV A) could be the dominant detection mechanism in relatively high impedance ($R > 100 \Omega$) point contacts, where probably $\beta_c > 1$. In this chapter it will be demonstrated in the PLL model for a current driven junction (dc and rf) that such a detection mode really exists and we show how it depends on various parameters. First, however, the behavior of the junction impedance will be represented under the same conditions, i.e., $\beta_c > 1$, but without hysteresis.

A. The impedance for $\beta_c > 1$

At a sufficiently high thermal noise level, a capacitively shunted junction with $\beta_c > 1$, has a nonhysteretic I - V curve. As shown in Sec. IV B the voltage across the junction consists of a series of pulse trains with average length τ_{out} separated by an average dead time τ_{in} (when the voltage across the junction is zero). When the junction is biased with $\bar{i}_{min} < \bar{i} < 1$, it is out of lock for a fraction of time

$$\bar{v}/\bar{v}_{out} = \tau_{out}/(\tau_{in} + \tau_{out}), \quad (23)$$

and in lock for the remaining fraction of time $1 - \bar{v}/\bar{v}_{out} = \tau_{in}/(\tau_{in} + \tau_{out})$. This means that the average rf impedance of the junction will be the sum of the in-lock impedance (exhibiting the plasma resonance) multiplied by a factor $1 - \bar{v}/\bar{v}_{out}$ and the out-of-lock impedance multiplied by a factor \bar{v}/\bar{v}_{out} .

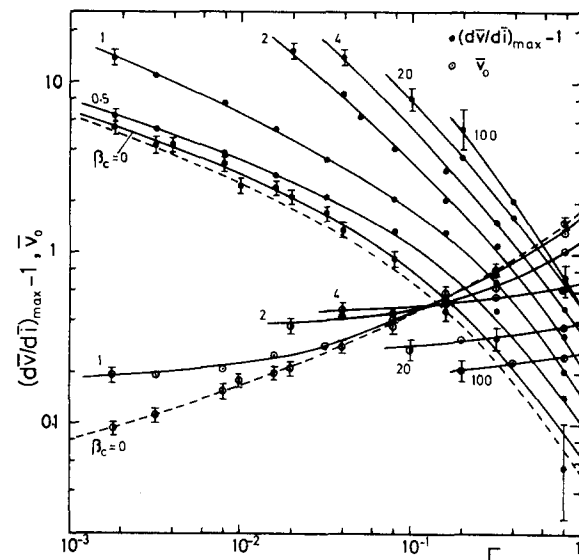


FIG. 9. Measured maximum value of the normalized excess differential resistance $[(d\bar{v}/d\bar{i})_{max} - 1]$ and the normalized average voltage \bar{v}_0 at which it occurs, both as a function of Γ and for several values of β_c . The accuracy in Γ is $\pm 10\%$. The dashed lines are calculated by Likharev *et al.*¹¹ for $\beta_c = 0$. The measured curve of \bar{v}_0 for $\beta_c = 0$ is the same as the dashed curve and therefore not shown separately.

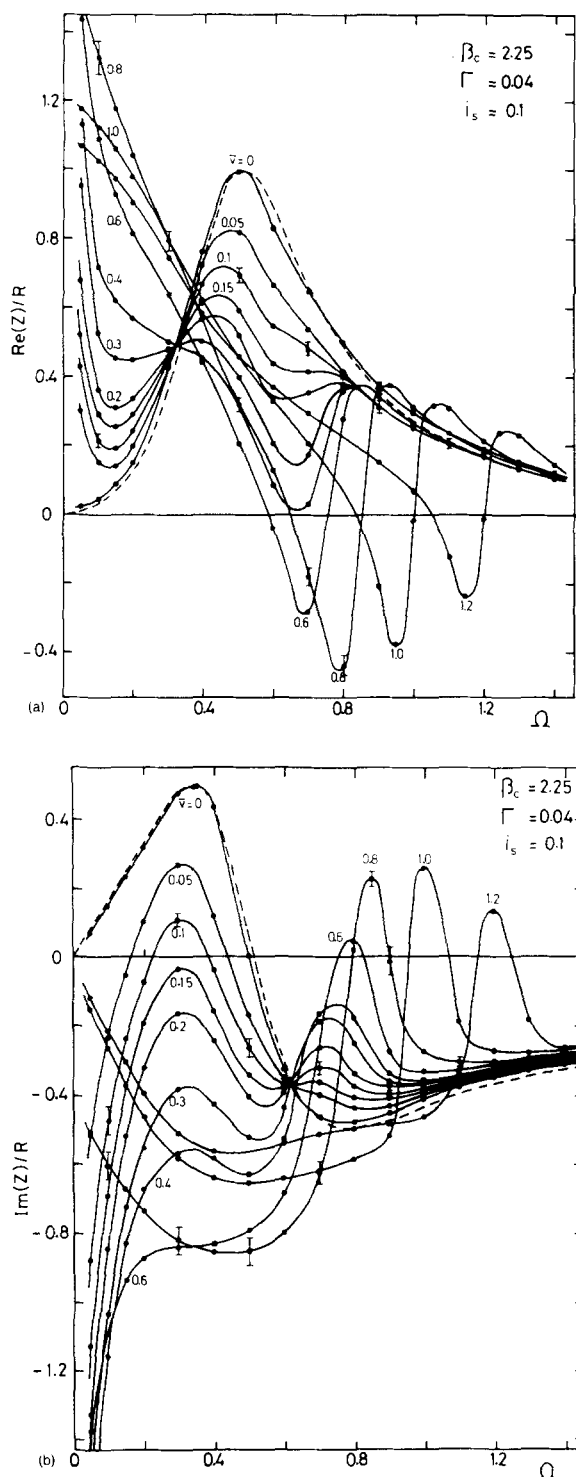


FIG. 10. Measured normalized real (a) and imaginary (b) part of the impedance of a capacitively shunted ($\beta_c = 2.25$) Josephson junction in the presence of thermal noise equivalent to $\Gamma = 0.04$, as a function of Ω and for several values of \bar{v} (compare Fig. 2 for $\beta_c = 0$). The dashed curves show the plasma resonance for $\bar{v} = 0$ and $\bar{i} = 0.75$ at $\Omega_p = (\cos\phi/\beta_c)^{1/2}$.

We used the PLL analog to measure the rf impedance as a function of frequency for two different β_c values. Fig. 10 shows $\text{Re}(z)$ and $\text{Im}(z)$ for $\beta_c = 2.25$ and $\Gamma = 0.04$ (as for $\beta_c = 0$). [The choice of this β_c value is determined by the later use of a chopping frequency in the case of wide-band detection. For $\Gamma = 0.04$, $\beta_c = 2.25$ is the largest β_c value

that can be used such that, for the voltage region of interest, τ_{in} and τ_{out} are still short (see Fig. 7) compared with one period of the chopping frequency].

For $\bar{v} = 0$ and $\bar{i} = 0.75$ the measured impedance is in good agreement with the calculated impedance when biased in the supercurrent (as shown by the dashed lines in Fig. 10), i.e., equal to a parallel combination of a resistance R , an inductance $L_J = \hbar/(2eI_c \cos\phi)^{-1}$, and a capacitance C , where ϕ is determined by the bias current according to $\phi = \arcsin \bar{i}$. Therefore the resonance that is observed at $\Omega = 0.50 \pm 0.02$ is completely described as a plasma resonance with $\Omega_p = (\cos\phi/\beta_c)^{1/2} = 0.54 \pm 0.02$. [When plotted in the complex plane, this is a full circle with the center at (0,0.5) and a radius of 0.5].

For $\bar{v} = 1.2$ the measured $\text{Re}(z)$ in Fig. 10(a) shows a smoothed singular impedance at $\Omega_{\text{sing}} = \bar{v} = 1.2$. However, the singularity is inverted with respect to the $\beta_c = 0$ case [see Fig. 2(a)]. In addition, the measured $\text{Im}(z)$ in Fig. 10(b) shows a positive peak centered at $\Omega = 1.2$, instead of the negative peak for $\beta_c = 0$ [see Fig. 2(b)]. These differences in behavior are in qualitative agreement with the impedance values calculated simply from the values for $\beta_c = 0$ (Fig. 2), when a capacitance is added in parallel [a complete description should also take into account the effect of the capacitive shunt at the idler frequency ($\Omega_i = 2\bar{v} - \Omega$)]. When \bar{v} is decreased, the singularity in $\text{Re}(z)$ occurs at $\Omega_{\text{sing}} = \bar{v}$ until $\bar{v} \approx 0.8$. For lower \bar{v} values, $\text{Re}(z)$ shows a smoothed singularity at $\Omega \approx 0.7$, for all \bar{v} values. It is decreasing in amplitude with decreasing \bar{v} . For $0.2 < \bar{v} < 0.6$ the bias current is approximately constant and at this bias current value, $\bar{v}_{\text{out}} \approx 0.7$ as is found by disconnecting the noise source. We therefore see that the singularity occurs at $\Omega = \bar{v}_{\text{out}}$. Since, upon decreasing \bar{v} , the junction is in lock for a growing fraction of the time, the plasma resonance increases in magnitude until, for $\bar{v} = 0$, only this plasma resonance remains. The resonance frequency for $\bar{v} \neq 0$ appears to be considerably lower than for $\bar{v} = 0$. This is partly caused by the influence of the negative branch of the smoothed singularity at $\Omega = \bar{v}_{\text{out}} \approx 0.7$. However, there is also a real decrease in the plasma resonance frequency, since, with increasing \bar{v} , also \bar{i} increases somewhat. For $\bar{v} = 0.3$ and $\bar{i} = 0.88$, e.g., $\Omega_p = (\cos\phi/\beta_c)^{1/2} = 0.46 \pm 0.02$, compared with $\Omega_p = 0.54 \pm 0.02$ for $\bar{v} = 0$ and $\bar{i} = 0.75$.

In addition to the plasma resonance peak and the smoothed singularity at $\Omega \approx 0.7$, we see from Fig. 10 that there is a special low-frequency effect. For $\Omega \leq 0.15$ and for $0.05 \leq \bar{v} < 0.8$, we see an increase in magnitude and a change in phase of the complex impedance, with respect to the weighted average of the in-lock and the out-of-lock impedance. When the measured values from Fig. 10 are plotted in the complex plane (not shown) we find that the phase shift is at a maximum of $60^\circ \pm 5^\circ$ for all values of $\bar{v} \leq 0.4$ and for $\Omega \leq 0.15$. This effect is caused by the switching of the junction between the phase-locked and the rotating-phase mode. It can be understood by considering the time-dependent behavior of the voltage across the junction, relative to the induced rf current. For $\Omega = 0.05$ and $\bar{v} = 0.1$, Fig. 11 shows the rf current through the junction (upper trace), a single track of the voltage across the junction (middle trace), and a

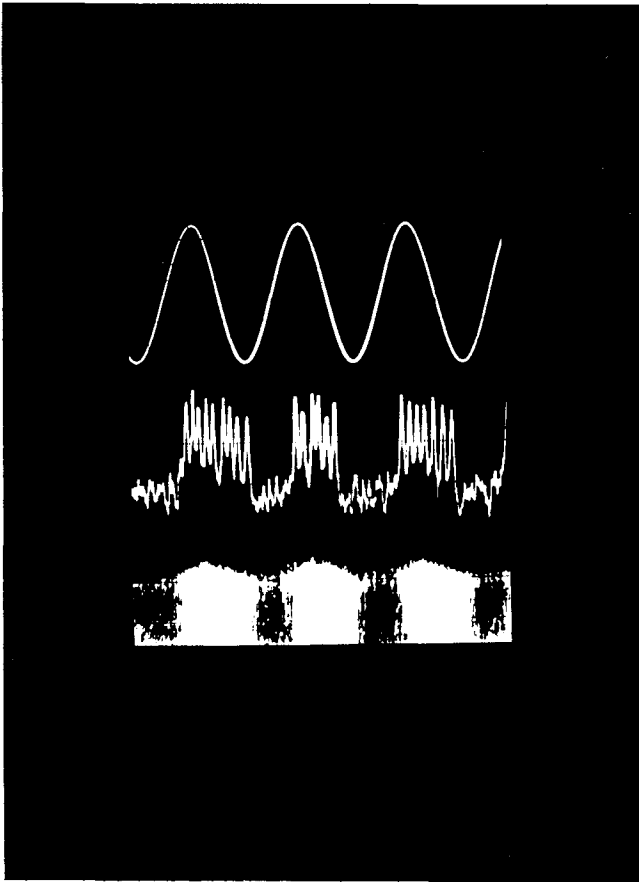


FIG. 11. rf current (upper trace) at $\Omega = 0.05$, through a junction with $\beta_c = 2.25$ and $\bar{v} = 0.1$, together with a single track of the voltage across the junction (middle trace), and a superposition of many tracks of the voltage across the junction (lower trace), as measured on an oscilloscope with the time base synchronized with the rf current.

superposition of many tracks of the voltage across the junction showing the time-averaged effect (lower trace), as measured on an oscilloscope with the time base synchronized to the rf current. The effect is caused by an enhanced chance for the system to be excited out of lock by thermal noise, when the rf current is near its maximum and an enhanced chance for the system to become phase locked again when the rf current is near its minimum. Apparently this effect is strongly frequency dependent and occurs only when $\Omega \ll 1$. In the mechanical analog (Fig. 6) low-frequency rf currents can be thought to modulate the slope of the washing board. When the rf current is near its maximum, the slope of the washing board is steeper than for \bar{i} so that the effective energy barrier is lower than for \bar{i} . This results in an enhanced chance for the particle to escape across the barrier by thermal noise. When the rf current is near its minimum, the slope of the washing board is less steep than for \bar{i} , so that the effective energy barrier is higher than for \bar{i} . This results in an enhanced chance for the particle to be recaptured in a potential well. Both effects therefore are synchronized with the rf signal, so that they contribute to the value of rf impedance. The rise time of the out-of-lock voltage pulses will limit the maximum frequency at which this effect can occur. For $\beta_c = 2.25$ this apparently occurs at $\Omega \simeq 0.15$.

The overall effect of this behavior is a large amplitude rf frequency component in the junction voltage (the amplitude of the rf pulses is \bar{v}_{out}). At the same time this frequency component has a phase shift with respect to the rf current.

The second β_c value used was 20 with a thermal noise level equivalent to $\Gamma = 0.2$. For $\beta_c = 20$ this is the smallest Γ value possible for which τ_{in} and τ_{out} are short enough to satisfy the condition mentioned for $\beta_c = 2.25$. The impedance, as shown in Fig. 12, is dominated by a resonance at $\Omega = 0.195 \pm 0.01$. This resonance frequency is in good agreement with the plasma resonance frequency that is measured without noise in the supercurrent ($\bar{v} = 0$) and for the same bias current, at $\Omega = 0.20 \pm 0.1$. Again there seems to be a small difference between the experimental resonance frequency and $\Omega_p = (\cos\phi/\beta_c)^{1/2} = 0.215 \pm 0.01$, for the same bias current. This may be caused by a nonperfect phase dependence of the pair current in the analog. However, the shape of $\text{Re}(z)$ and $\text{Im}(z)$ perfectly match an L, C resonance, (as for $\beta_c = 2.25$), indicating that indeed a resonance occurs at the plasma frequency. The Q value of the plasma resonance⁶ is $Q = \omega_p RC = (\beta_c \cos\phi)^{1/2} = \Omega_p \beta_c = 4.3 \pm 0.2$, in good agreement with the experimental value $Q_{\text{exp}} = 4.0 \pm 0.4$.

For $\bar{v} = 0.6$ the junction is permanently out-of-lock (as can be seen on the oscilloscope), so that the impedance is entirely the out of lock impedance. Because at this β_c value the junction oscillations, when out of lock, are purely sinu-

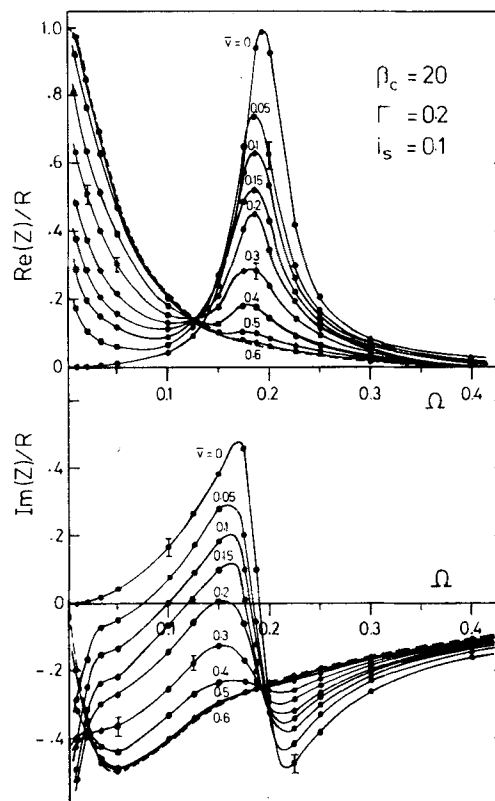


FIG. 12. Measured normalized real and imaginary part of the impedance of a capacitively shunted junction ($\beta_c = 20$) in the presence of thermal noise ($\Gamma = 0.2$), as a function of Ω and for several values of \bar{v} .

soidal (all harmonics are effectively shunted by the capacitance), the impedance will be equal to the impedance of the parallel combination of R and C . It is given by the dashed lines in Fig. 12 and is in good agreement with the experimental values for $\bar{v} = 0.6$. In the complex plane this impedance is a semicircle in the fourth quadrant, with the center at $(0, 0.5)$ and a radius of 0.5. For different values of \bar{v} and for $\Omega \gtrsim 0.05$ the impedance is again rather well described by the sum of the fractions $\bar{v}/\bar{v}_{\text{out}}$ and $1 - \bar{v}/\bar{v}_{\text{out}}$ of the out-of-lock and the in-lock impedance, respectively. The deviation from this sum, at low frequencies, is much smaller than for $\beta_c = 2.25$. For increasing \bar{v} the bias current also increases somewhat, so that the plasma frequency [through $\Omega_p = (\cos\phi/\beta_c)^{1/2}$] decreases slightly from the $\bar{v} = 0$ frequency (see Fig. 12). The decrease is much smaller than for $\beta_c = 2.25$ because \bar{i} is almost constant for $0.05 \leq \bar{v} \leq 0.5$.

From the measurements it is clear that the average rf impedance of a capacitively shunted RSJ model junction in the presence of thermal noise indeed shows very clearly a plasma resonance even for a nonzero average voltage across the junction. The observed resonance frequency is in good agreement with the theoretical plasma resonance frequency in the supercurrent.

B. The wide-band response for $\beta_c > 1$

Next we investigated the effects of the plasma resonance on the detection properties of underdamped ($\beta_c \gtrsim 1$) RSJ model Josephson junctions. With the PLL analog the wide-band response to radiation was measured for several β_c values. We expect a plasma resonance in the response, resulting from the junction being thrown out of lock, while a smoothed singular response will occur when the junction is out of lock.

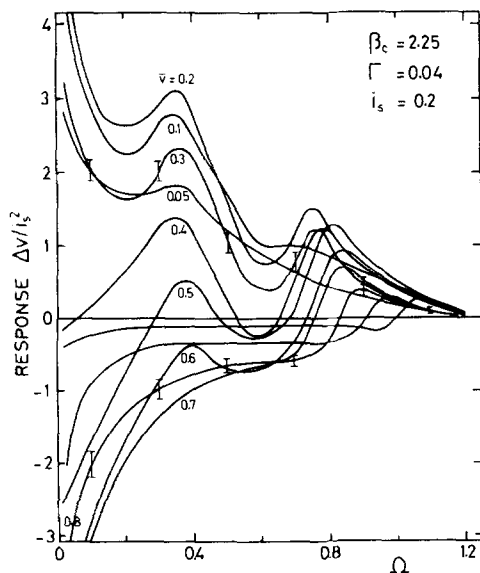


FIG. 13. Measured normalized response to radiation of a capacitively shunted junction ($\beta_c = 2.25$) as a function of frequency, for several values of \bar{v} .

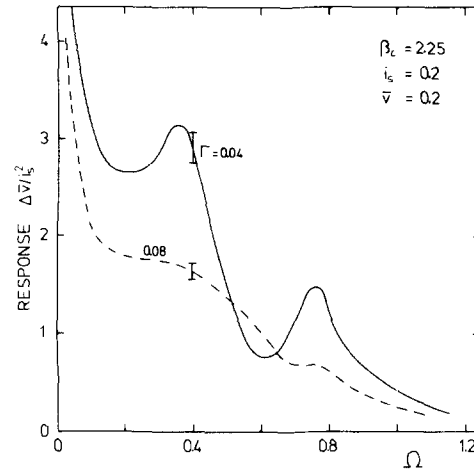


FIG. 14. Comparison between the measured normalized responses for two different thermal noise levels, as a function of Ω , for $\beta_c = 2.25$, and $\bar{v} = 0.2$.

Figure 13 gives the response $\Delta\bar{v}/i_s^2$ to an rf current (with amplitude i_s) as a function of the rf frequency for several values of \bar{v} , in the case $\beta_c = 2.25$ and $\Gamma = 0.04$. For $\bar{v} = 1.0$ the response consists of a smoothed singular response at a frequency $\Omega_{\text{sing}} = \bar{v}$. For decreasing \bar{v} the value of Ω_{sing} decreases until $\Omega_{\text{sing}} \simeq \bar{v} = 0.8$. Upon a further decrease of \bar{v} , Ω_{sing} is no longer equal to \bar{v} but instead remains at $\simeq 0.7$, which is approximately equal to \bar{v}_{out} , as observed for the impedance. At the same time an enhanced response develops for $\Omega = 0.35 \pm 0.03$. This resonance frequency is significantly below the plasma resonance frequency that was measured in the impedance at $\Omega = 0.50 \pm 0.02$. In order to obtain a reasonable signal-to-noise ratio a signal current $i_s = 0.2$ was required. (The rf impedance for $i_s = 0.2$ is not different from the measurements with $i_s = 0.1$.) The difference in resonance frequency is not caused by the larger rf current amplitude. For $i_s = 0.2$ a reduction of the plasma resonance frequency of about one percent would be expected.⁶ Predominantly at low frequencies, a third contribution to the response is observed. It is positive for $\bar{v} < 0.4$ and negative for $\bar{v} > 0.4$ and approximately follows $(d^2\bar{v}/d\bar{i}^2)$, with an additional RC cutoff as a function of frequency. Therefore this is a classical response, analogous to the classical response for $\beta_c = 0$.¹⁸

At $\beta_c = 2.25$, both the resonance in the response and the smoothed singular response are easily destroyed by increasing the noise level. Figure 14 shows the response for $\beta_c = 2.25$ and $\bar{v} = 0.2$, for $\Gamma = 0.04$ and $\Gamma = 0.08$, respectively. We see that at $\Gamma = 0.08$ both features on the response have almost disappeared. For $\Gamma < 0.04$ the resonance probably increases in amplitude but this requires too large integration times to be measured with the analog. Saturation of the resonance occurs for signal currents larger than $i_s = 0.2$.

Again β_c was increased to 20 with a thermal noise level equivalent to $\Gamma = 0.2$. The response is shown in Fig. 15 as a function of frequency. It is dominated by a resonance at $\Omega = 0.175 \pm 0.01$, which must be compared with the plasma frequency at $\Omega_p = (\cos\phi/\beta_c)^{1/2} = 0.215 \pm 0.01$ for the same \bar{i} . The Q factor of the resonance is considerably smaller than

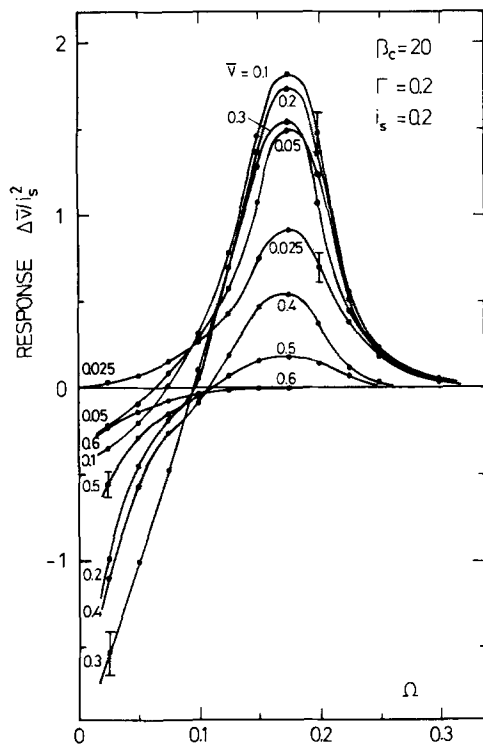


FIG. 15. Measured normalized response vs Ω , of a junction with $\beta_c = 20$ and $\Gamma = 0.2$, for several values of \bar{v} .

for the impedance measurement. The smoothed singular response at $\Omega = \bar{v}_{out}$ has fully disappeared in this case. An unexpected phenomenon is the occurrence of a negative response at low frequencies for all values of \bar{v} ; apparently it is approximately proportional to $(-d\bar{v}/d\bar{i})$ contrary to the case of $\beta_c = 2.25$. Figure 16 gives the response for $\beta_c = 20$ at the resonance frequency and for the voltage of maximum response ($\bar{v} = 0.1$) as a function of the noise level. The solid line represents the response when measured with a chopping frequency $\Omega_{chop} = 5 \times 10^{-3}$ (as for all measurements up to now). When Γ is decreased, the response first increases. This

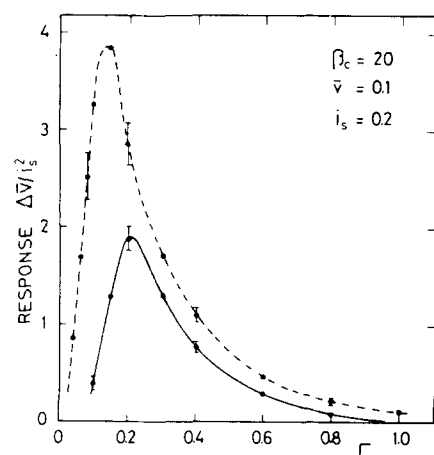


FIG. 16. Measured normalized response vs Γ , for $\beta_c = 20$ and $\bar{v} = 0.1$. The solid line is for a chopping frequency $\Omega_{chop} = 5 \times 10^{-3}$ and the dashed line is for $\Omega_{chop} = 5 \times 10^{-4}$.

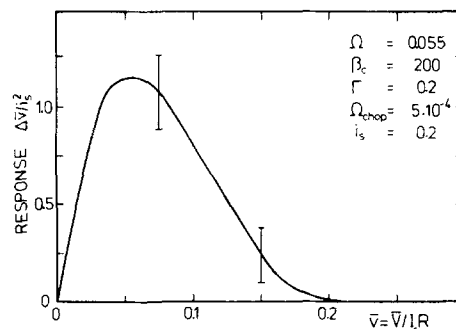


FIG. 17. Average curve, drawn through a trace (not shown) of the measured normalized response vs \bar{v} , at the frequency of maximum response, for a junction with $\beta_c = 200$ and $\Gamma = 0.2$.

is caused by both an increase of the differential resistance $d\bar{v}/d\bar{i}$ and an increase of the Q factor of the resonance. Also, upon decreasing Γ , τ_{in} , and τ_{out} increase (see Sec. IV A) for the same value of \bar{i} . When τ_{in} and τ_{out} become of the same order of magnitude as the period of the chopping frequency Ω_{chop} , at which the rf current is modulated, the response decreases dramatically. The resulting maximum occurs at $\Gamma \equiv \Gamma_{crit}$. The dashed line in Fig. 16 is for a chopping frequency $\Omega_{chop} = 5 \times 10^{-4}$. In this case Γ_{crit} is lower than for $\Omega_{chop} = 5 \times 10^{-3}$, as expected. For higher Γ values the response for $\Omega_{chop} = 5 \times 10^{-4}$ is also higher than for $\Omega_{chop} = 5 \times 10^{-3}$. This effect is not understood. (For $\beta_c = 2.25$, a reduction of the chopping frequency below $\Omega_{chop} = 5 \times 10^{-3}$ does not change the response for $\Gamma > \Gamma_{crit}$; only Γ_{crit} is lowered somewhat, as expected.) By comparing Fig. 16 with Fig. 5 we find that for the same Γ value ($\Gamma = 0.2$) the response for $\beta_c = 20$ at the resonance frequency is approximately three times as large (in an absolute sense) as the response for $\beta_c = 0$ at the same frequency ($\Omega_{chop} = 5 \times 10^{-4}$).

Even for $\beta_c = 200$ we measured an enhanced response at $\Omega = 0.055 \pm 0.01$ compared with a calculated value of the plasma frequency of $\Omega_p = (\cos\phi/\beta_c)^{1/2} = 0.070 \pm 0.004$. Because of the long average lifetimes in this case, the chopping frequency $\Omega_{chop} = 5 \times 10^{-4}$ is barely sufficient to detect this response. It is given in Fig. 17 as a function of \bar{v} , for the same Γ value as for $\beta_c = 20$ ($\Gamma = 0.2$). In Fig. 18 the response at the resonance frequency for $\bar{v} = 0.05$ is given as a function of Γ . Again the response decreases below Γ_{crit} ,

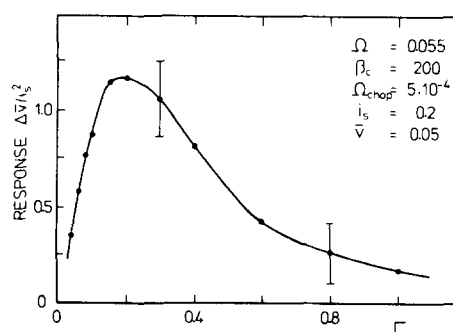


FIG. 18. Measured normalized response vs Γ , at the voltage and frequency for maximum response, for $\beta_c = 200$.

which is somewhat higher in this case than for $\beta_c = 20$ at the same chopping frequency, as expected.

In a real Josephson junction there is also a limiting value of Γ . However, this value will be lower than Γ_{crit} in the analog for the same β_c value because the normalized chopping frequency is 10^{-7} – 10^{-8} times the value in the analog. The limiting value of Γ in a real Josephson junction with $\beta_c = 200$ is approximately 0.05 [as can be found using Eqs. (19) and (21)] as compared with $\Gamma_{\text{crit}} \approx 0.2$ for the analog (Fig. 18). The resonant response for $\beta_c = 200$ and $\Gamma = 0.2$ is of the same size as the response for $\beta_c = 0$ at the same frequency and the same noise level (see Fig. 5). In view of the effect of the chopping frequency on the amplitude of the response, as found for $\beta_c = 20$ (Fig. 16), the real response for $\beta_c = 200$ may be higher than for $\beta_c = 0$.

C. Interpretation of the response measurements

In a capacitively shunted junction with an I - V curve that is nonhysteretic by the influence of thermal noise, several wide-band response mechanisms can occur. When the junction is biased such that $\bar{i}_{\text{min}} < \bar{i} < 1$ (\bar{i}_{min} ; see Fig. 8), the average normalized voltage across the junction is given by Eq. (23). This means that a response to an rf current will take place, when τ_{in} and/or τ_{out} are influenced by the rf current and also when the out-of-lock I - V curve changes, owing to the rf current, resulting in a different \bar{v}_{out} .

The influence of the rf current on the shape of the out-of-lock branch of the I - V curve will be similar to the $\beta_c = 0$ case. The internal oscillations of the junction will try to lock on the rf frequency, resulting in a (smoothed) singular response around $\bar{v}_{\text{out}} = \Omega$. However, the amplitude of the response is much smaller than for $\beta_c = 0$, because the rf signal and the internal oscillations are shunted by the capacitance. With increasing frequency and β_c , their amplitudes decrease.

The influence of the rf current on τ_{in} is twofold. First there is the reduction of the effective energy barrier E_B (see Fig. 6 and Sec. IV A) that also takes place when $\beta_c = 0$. This results in an enhanced chance for the junction to be excited out of lock, reducing τ_{in} . In addition, there is a resonant reduction of τ_{in} by an rf current with a frequency equal to the attempt frequency, with which the phase ϕ attempts to escape from the phase-locked mode. In terms of the mechanical analog (Fig. 6), the particle attempts to escape from the potential well, by thermal noise, at a normalized attempt frequency Ω_a [see Eq. (17)]. This means that when the slope of the washing board is modulated at this attempt frequency, an enhanced excitation out of the potential well is expected, thus further reducing τ_{in} . This results in an enhanced wide-band response when the rf current has the frequency Ω_a .

The following is a qualitative picture of the attempt frequency, using the mechanical analog. Owing to low-frequency components in the thermal noise, the slope i of the washing board slowly changes around \bar{i} . The resonance frequency of the particle at the bottom of the potential well (the plasma frequency) decreases for $i > \bar{i}$, because of a decreasing curvature of the potential well [$\Omega_p = (\cos\phi/\beta_c)^{1/2}$]. The chance

that the particle is excited out of the potential well by the high-frequency part of the thermal noise (that contains almost the total thermal noise power) together with the modulation of the slope of the washing board at the rf frequency increases with increasing i (since the energy barrier becomes smaller). This means that the effective resonance occurs at a frequency corresponding to a certain \bar{i}_{exc} , from where, on the average, the particle is excited. Since this \bar{i}_{exc} is larger than \bar{i} (see Fig. 19), the effective resonance frequency for excitation out of the potential well (the attempt frequency) is lower than the resonance frequency corresponding to \bar{i} . In a real junction this means that the effective plasma resonance frequency (the attempt frequency) is below the plasma frequency for \bar{i} . In the limit of small β_c values, where \bar{i}_{min} is close to one, there is a substantial difference between Ω_a and Ω_p , because $\cos\phi$ depends very strongly on \bar{i} in this region (see Fig. 19). For large β_c values, however, $\cos\phi$ hardly depends on \bar{i} , in the region of interest, so that Ω_a and Ω_p are almost equal (see also Fig. 20).

The influence of the rf current on τ_{out} probably only occurs by a reduction of the effective energy barrier against recapture (see Sec. IV A), similar to the case of τ_{in} . A resonant excitation across the energy barrier against recapture, however, might be possible. This would result in a negative response around the attempt frequency (assumed to be) $\Omega = \bar{v}_{\text{out}}$ in this case.

When the effects of a rf current on the average lifetimes and on \bar{v}_{out} are taken together, the following wide-band response may be expected. The smoothed singular response at \bar{v}_{out} will be present for all values of \bar{v} when β_c is not too large. The position of the singularity will be at $\Omega = \bar{v}_{\text{out}}$ and the amplitude will be proportional to $\bar{v}/\bar{v}_{\text{out}}$ (the fraction of time that the junction is out of lock). This means that even for $\bar{v} \ll \bar{v}_{\text{out}}$ there will be a very small amplitude, smoothed singular response at $\Omega = \bar{v}_{\text{out}}$. Further, there will be a positive response to a rf current due to a decrease of τ_{in} and a negative response due to a decrease of τ_{out} . Since the fraction of time that the junction is in lock, or out of lock is given by $1 - \bar{v}/\bar{v}_{\text{out}} = \tau_{\text{in}}/(\tau_{\text{in}} + \tau_{\text{out}})$ and $\bar{v}/\bar{v}_{\text{out}} = \tau_{\text{out}}/(\tau_{\text{in}} + \tau_{\text{out}})$, respectively, a response behavior may be expected, that is symmetrical with respect to $\bar{v} = i/2$ (where $\tau_{\text{in}} = \tau_{\text{out}}$); i.e., a positive response for $\bar{v} < i/2$ and a negative response for $\bar{v} > i/2$. The response that results from

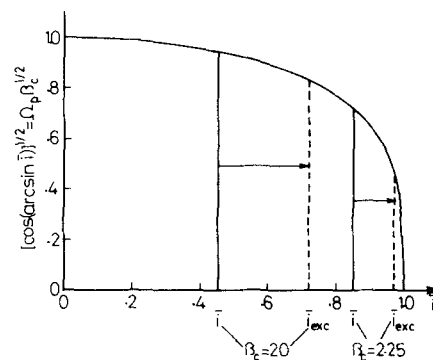


FIG. 19. Schematic drawing, illustrating the qualitative effect of noise on \bar{i}_{exc} and thereby on the resonance frequency in the response for $\beta_c = 2.25$ and $\beta_c = 20$.

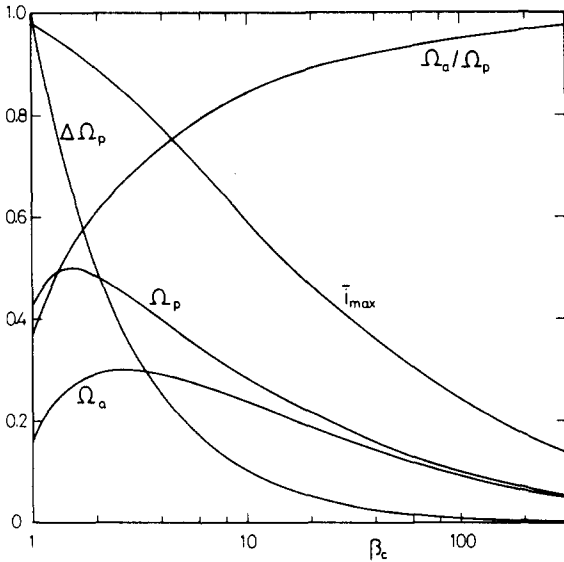


FIG. 20. Bias current \bar{i}_{\max} for maximum response in case of small Γ values, as a function of β_c . Also shown are the corresponding values of the plasma frequency Ω_p , the attempt frequency Ω_a , the ratio Ω_a/Ω_p , and the width $\Delta\Omega_p$ of the plasma resonance.

a reduction of E_B (Josephson response $\sim d\bar{v}/d\bar{i}$) would fall off with frequency as Ω^{-2} in the case $\beta_c = 0$; with a capacitive shunt, the frequency dependence is expected to be even stronger than that. The classical response $d^2\bar{v}/d\bar{i}^2$ is not frequency dependent for $\beta_c = 0$, so for $\beta_c \neq 0$ we expect a simple RC falloff.

The decrease of τ_{in} for rf frequencies around Ω_a results in a positive resonant response. This response will be zero for $\bar{v} = 0$ and increases for increasing \bar{v} . For $\bar{v} > \bar{v}_{out}/2$ it will decrease again, because of a decreasing τ_{in} . When the junction is almost permanently out of lock ($\bar{v} \simeq \bar{v}_{out}$), this response mode will have disappeared.

The wide-band response, measured with the PLL analog (Sec. V B) can be qualitatively described by the different response mechanisms introduced above. For $\beta_c = 2.25$, the smoothed singular response indeed occurs for $\Omega = \bar{v}_{out} \simeq 0.7$ (for $0.1 < \bar{v} < 0.6$, \bar{i} is about constant and $\bar{v}_{out} \simeq 0.7$) and decreases in magnitude for decreasing \bar{v} , below $\bar{v} = 0.7$. The response that is approximately proportional to $(d^2\bar{v}/d\bar{i}^2)$ in the measurements, is indeed positive when $\bar{v} < \bar{v}_{out}/2$, with a maximum for small \bar{v} , and negative when $\bar{v} > \bar{v}_{out}/2$, with a maximum for $\bar{v} \simeq 3\bar{v}_{out}/4$. The frequency dependence is not as strong as expected, however (less than Ω^{-1} instead of an expected dependence $\sim \Omega^{-2}$). The enhanced response at a resonance frequency of $\Omega = 0.35 \pm 0.03$ in the measurements is in rather good agreement with the attempt frequency $\Omega_a = 0.31 \pm 0.01$ [given by Eq. (17)], and differs considerably from the plasma frequency $\Omega_p = 0.48 \pm 0.02$ (for $\bar{i} = 0.85$ with $\bar{v} \simeq 0.15$).

For $\beta_c = 20$, the measurements show that the (smoothed) singular response has disappeared completely. This is caused by the large capacitive shunt. The resonance frequency in the response at $\Omega = 0.175 \pm 0.01$ is somewhat below the attempt frequency $\Omega_a = 0.19 \pm 0.01$ as given by Eq. (17). However, the plasma resonance frequency in the

impedance is also somewhat below $\Omega_p = (\cos\phi/\beta_c)^{1/2}$. The ratio of the resonance frequency in the response and the resonance frequency in the impedance as found from the measurements, 0.90 ± 0.06 , is in good agreement with the calculated ratio $\Omega_a/\Omega_p = 0.88 \pm 0.06$. The behavior of the amplitude of the resonance is similar to that for $\beta_c = 2.25$. The negative response for small Ω may be caused by the reduction of τ_{out} by the rf current.

For $\beta_c = 200$ an enhancement of the response takes place at $\Omega = 0.055 \pm 0.01$, while the attempt frequency is equal to $\Omega_a = 0.068 \pm 0.004$, which is almost equal to the plasma frequency $\Omega_p = 0.070 \pm 0.004$ (see also Fig. 20). The signal-to-noise ratio is so poor that it is not useful to present a graph of the response as a function of frequency for different bias voltages. The response at $\Omega = 0.055$, however, depends on \bar{v} and Γ in the same way as for smaller β_c values (see Figs. 17 and 18).

From the measurements we can draw the conclusion that for a capacitively shunted junction in the presence of thermal noise and nonzero average voltage a response mechanism exists that causes an enhancement in the wide-band response at a frequency that is in rather good agreement with the attempt frequency. The fact that the junction noise will be enhanced by the switching between the in-lock and the out-of-lock mode, however, means that the signal-to-noise ratio probably does not improve from the $\beta_c = 0$ case.

D. The effect of an external system on the plasma resonance

The measurements of the rf impedance and the wide-band response of the capacitively shunted RSJ model described above have been made for a current biased rf junction. When the junction is connected to an external system, the response will depend on the rf impedance of the external system, on the rf impedance of the junction, and on the rf current response of the junction. Since the rf impedance of the junction will be modified by the external system, the junction impedance as measured in Sec. V A will not be correct in this case. This situation is rather complex. In order to gain some insight into the problem the modification of the impedance is ignored. The influence of the plasma resonance on the system response can then be qualitatively described as follows. Both the impedance and the response can be separated in an in-lock part that exhibits the plasma resonance and in an out-of-lock part. It is obvious that, when the impedance and the response are combined, the in-lock impedance should be combined with the in-lock response, and the out-of-lock impedance with the out-of-lock response. This results in a combination of the plasma resonance and the resonance at the attempt frequency.

Fig. 20 gives the normalized values of Ω_a and Ω_p as a function of β_c , using Eqs. (17) and (22), respectively. The value of $\cos\phi$ in these equations is obtained by using the curve for \bar{i}_{\max} as a function of β_c , where \bar{i}_{\max} is the bias current for maximum response (approximately \bar{i} where $\bar{v} = \bar{v}_{out}/2$) obtained from the analog for small Γ values. Also given in Fig. 20 is the ratio Ω_a/Ω_p (that decreases for decreasing β_c) and the bandwidth $\Delta\Omega_p$ of the plasma reso-

nance, where

$$Q = \Omega_p / \Delta\Omega_p = \omega_p RC = \Omega_p \beta_c, \quad (24)$$

giving

$$\Delta\Omega_p = \beta_c^{-1} \quad (25)$$

From Fig. 20 it is clear that $\Delta\Omega_p > (\Omega_p - \Omega_a)$, while the resonance in the response is even broader than in the impedance. This implies that in a junction coupled to an external system it is not possible to discriminate between the plasma resonance and the resonance at the attempt frequency. Instead there will be one broader maximum, somewhere between Ω_a and Ω_p .

The measurements by Tolner⁷ show that the response of a junction (with an estimated $\beta_c > 1$) in a free standing antenna system exhibits a resonance frequency that changes with the normal-state resistance R of the contact. Later measurements by us on point contacts in a waveguide²² show a similar behavior of the amplitude of sharp structural resonances, as a function of R . These measurements^{7,22} therefore suggest that, also in Josephson junctions in an external system, a resonant enhancement of the response, caused by plasma effects, exists.

VI. CONCLUSIONS

The rf impedance and the wide-band response to radiation of a Josephson junction in the presence of thermal noise have been measured with a phase-locked-loop analog. Both the situation with and without a capacitive shunt were investigated. For $\beta_c = 0$ both the rf impedance and the response were found to be in good agreement with existing theory.

For a capacitive shunt, such that the noiseless I - V curve is hysteretic, the average lifetimes of the junction in lock τ_{in} ($\bar{v} = 0$) and out of lock τ_{out} ($\bar{v} = \bar{v}_{out}$) were measured in the presence of thermal noise. τ_{in} is in good agreement with existing theory²⁰ and the behavior of τ_{out} is derived from the measurements. This means that, although a junction with $\beta_c > 1$ seems single valued when a sufficiently large thermal noise level is present, (such that the time constant with which the voltage is measured is long compared with τ_{in} and τ_{out}), it actually is either in lock or completely out of lock.

The rf impedance and the response to radiation of an underdamped Josephson junction ($\beta_c > 1$) in the presence of thermal noise show very clearly the influence of plasma effects for *nonzero* average voltages across the junction. This is caused by the fact that, although $\bar{v} \neq 0$, the junction is in the zero voltage state for a fraction $\tau_{in}/(\tau_{in} + \tau_{out})$ of the time. The rf impedance shows a resonance at the plasma frequen-

cy. For $\beta_c = 20$ this plasma resonance is the dominating feature in the impedance. The response shows a resonant enhancement at a frequency below the plasma frequency that is in rather good agreement with the attempt frequency Ω_a with which the phase attempts to escape from the phase-locked mode ($\bar{v} = 0$) by thermal noise. For $\beta_c = 2.25$ a resonant response is found close to the attempt frequency, together with a smoothed singular response at $\Omega_{sing} = \bar{v}_{out}$ instead of at \bar{v} . For $\beta_c = 20$ the smoothed singular response has disappeared owing to the capacitive cutoff and only a resonance, near the attempt frequency, is present. The maximum response at this resonance is about three times as large as the response in the case $\beta_c = 0$ for the same frequency and noise level. Even for $\beta_c = 200$ a resonant response is observed that is of about the same size as the $\beta_c = 0$ response for the same frequency and noise level.

In conclusion we can say that, although a capacitive shunt deteriorates the normal wide-band Josephson response, a plasma induced response occurs that can even be larger than the response for $\beta_c = 0$.

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